

A proof that $\sqrt{2}$ is irrational

Here is a proof that there is no rational number whose square is 2.

The idea of the proof is to suppose that there *is* a rational number whose square is 2. You'll see that this leads to consequences that cannot be true. This means that there cannot, in fact, be any such number.

The proof depends on the following property of odd numbers: if you square an odd number, then the result is also odd. For example, $3^2 = 9$, $5^2 = 25$, $7^2 = 49$, and so on.

So, suppose that there is a rational number whose square is 2. Because it is rational, it can be written in the form

$$\frac{\text{integer}}{\text{integer}}.$$

Let's cancel this fraction down to its simplest form, say

$$\frac{m}{n}.$$

Since this fraction is in its simplest form, the integers m and n have no common factors.

Because

$$\left(\frac{m}{n}\right)^2 = 2,$$

it follows that

$$\frac{m^2}{n^2} = 2;$$

that is,

$$m^2 = 2n^2. \tag{1}$$

The number $2n^2$ is even, because it is the product of 2 and a whole number. Hence m^2 is even. Now if m were odd, then m^2 would also be odd; so, since m^2 is even, m must be even. This means that $m = 2r$ for some integer r . If we substitute $m = 2r$ into equation (1) then we obtain

$$(2r)^2 = 2n^2; \quad \text{that is,} \quad 4r^2 = 2n^2.$$

It follows that

$$2r^2 = n^2.$$

Now you can see that the integer n^2 is even, so n is also even.

So both m and n are even. But this is impossible, because m and n have no common factors!

So if there is a rational number whose square is 2, then there are consequences that are clearly not true. It follows that there cannot be any such rational number.

This proof is one of the best-known examples of a type of proof known as **proof by contradiction**, where you prove that a statement is true by showing that if it is false then there are consequences that are clearly not true. If you go on to study the module *Essential Mathematics 2* (MST125), then you will learn more about different ways of proving mathematical statements.

You may have wondered why it is true that, when you square an odd number, the result is always odd. Any odd number can be written in the form $2r + 1$ where r is an integer, so it can be represented by a row of $2r + 1$ dots. When this number is squared, the result is represented by a $2r + 1$ by $2r + 1$ square of dots, as shown in Figure 1. You can see that if the dots are divided up as shown in the figure, then there is an even number of dots in each region, except for the region in the bottom right-hand corner, where there is just one dot. So the total number of dots is the sum of three even numbers plus one, and hence is odd.

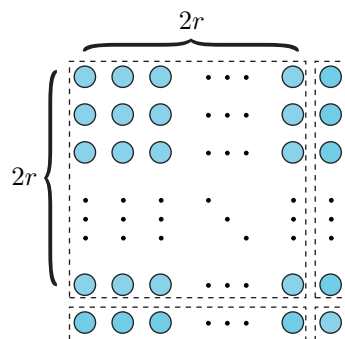


Figure 1 A square of dots representing the number $(2r + 1)^2$